Abstracts

Nonlinear and Nonlocal Degenerate Diffusions on Bounded Domains

Matteo Bonforte
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We study quantitative properties of nonnegative solutions to a nonlinear and nonlocal diffusion equation of the form $u_t = \mathcal{L}F(u)$ posed in a bounded domain, with appropriate homogeneous Dirichlet boundary conditions. The diffusion is driven by a linear operator $\mathcal{L}$ in a quite general class, that includes the three most common versions of the fractional Laplacians on a bounded domain with zero Dirichlet boundary conditions, as well as many other examples. The nonlinearity F is allowed to be degenerate, the prototype being $F(u) = |u|^{m-1}u$, with $m > 1$.

We will shortly present some recent results about existence, uniqueness and a priori estimates for a quite large class of very weak solutions, that we call weak dual solutions.

Then we will concentrate on the regularity theory: space time decay and positivity, boundary behavior, Harnack inequalities, interior and boundary regularity, and asymptotic behavior. All this is done in a quantitative way, based on sharp a priori estimates. Although our focus is on the fractional models, our techniques cover also the local case $s = 1$ and provide new results even in this setting.

A surprising instance of this problem is the possible presence of non-matching powers for the boundary behavior: this unexpected phenomenon is a completely new feature of the nonlocal and nonlinear structure of this model, and it is not present in the semilinear elliptic case, for which we will shortly present the most recent results.

The above results are contained in a series of recent papers in collaboration with A. Figalli (ETH Zürich, CH), Y. Sire (J. Hopkins Univ. Baltimore, USA), X. Ros-Otón (Univ. Zürich, CH) and J. L. Vázquez (UAM Madrid, ES).

On total variation flow under dynamic boundary conditions

Yoshikazu Giga
The University of Tokyo

We consider a total variation flow under dynamic boundary conditions. Such a problem appears in materials science as a prototype model. We formulate this problem as a gradient flow of a convex, lower semicontinuous energy in a Hilbert space. This enables us to construct a unique globally-in-time solution. We are also able to calculate the speed of a flat portion called a facet. However, there is a chance that boundary detachment phenomena occur. We introduce the notion of coherency which prevents boundary detachment phenomena if the facet is calibrable. The boundary detachment depends on mean-curvature of the boundary. It is expected that the boundary detachment does not occur when the boundary is mean-convex. We give several examples of radial case to support this conjecture. This is a joint work with R. Nakayashiki (Chiba), P. Rybka (Warsaw) and K. Shirakawa (Chiba).

Exponential stability of traveling waves for a constrained Allen-Cahn equation

Goro Akagi
Tohoku University

This talk concerns a one-dimensional Allen-Cahn equation on the whole line with the positive-part function, which constrains the growth of each solution to be non-decreasing. The study on such constrained evolution equations are motivated from phase-field models in Damage Mechanics. Indeed, evolution of damage is supposed to be monotone due to the nature of damaging. In this talk, we shall discuss traveling wave dynamics, which has been well studied for classical Allen-Cahn equations, for the constrained one. More precisely, we shall start with constructing a one-parameter family of *degenerate* traveling wave solutions (identified when coinciding up to translation) and investigate their properties. Furthermore, the traveling wave dynamics turns out to be relevant to a free boundary problem with a peculiar motion equation for the boundary through an analysis on a regularity issue for the constrained Allen-Cahn equation, and then, such a viewpoint enables us to prove exponential stability of degenerate traveling waves with some basin of attraction, although they are unstable in a usual sense. This talk is based on a joint work with C. Kuehn (München) and K.-I. Nakamura (Kanazawa).

Existence of solutions for an inhomogeneous fractional semilinear heat equation

Kotaro Hisa
Tohoku University

In this talk, we consider necessary conditions and sufficient conditions on the inhomogeneous term for the solvability of the Cauchy problem

$$(\mathrm{P}) \qquad \partial_t u + (-\Delta)^\alpha u = u^p + \mu, \qquad x \in \mathbf{R}^N,\ \ t > 0, \qquad u(0) = 0 \quad \text{in} \quad \mathbf{R}^N,$$

where $N \geq 1$, $0 < \alpha \leq 1$, $p > 1$ and μ is a Radon measure or a measurable function in $\mathbf{R}^N$.

The existence and nonexistence of the global-in-time solution to problem (P) have been already obtained by Bernrd [1], Zhang [3] and so on. However, there are few results on the local existence even in the case of $\alpha = 1$. The main result of this talk is concerned with the necessary condition for the local existence.

Theorem A ([2]). *Let u be a solution of (P) in $\mathbf{R}^N \times (0,T)$, where $0 < T < \infty$. Then there exists $\gamma_1 > 0$ depending only on N, α and p such that*

$$\sup_{x \in \mathbf{R}^N} \mu(B(x,\sigma)) \leq \begin{cases} \gamma_1 \sigma^{N - \frac{2p\alpha}{p-1}} & (p \neq p_\alpha), \\ \gamma_1 \left[\log \left(e + \dfrac{T^{\frac{1}{2\alpha}}}{\sigma} \right) \right]^{-\frac{N}{2\alpha}+1} & (p = p_\alpha), \end{cases}$$

for $0 < \sigma \le T^{\frac{1}{2\alpha}}$. *Here* p_α *is defined by*

$$p_\alpha := \frac{N}{N-2\alpha} \quad \textit{if} \quad 0 < 2\alpha < N \quad \textit{and} \quad p_\alpha := \infty \quad \textit{if} \quad 2\alpha \ge N.$$

From Theorem A we observe the following: There exists $C_* > 0$ such that if

$$\mu(x) \ge \begin{cases} C|x|^{-\frac{2p\alpha}{p-1}} & (p > p_\alpha), \\ C|x|^{-N}\left[\log\left(e + \dfrac{1}{|x|}\right)\right]^{-\frac{N}{2\alpha}} & (p = p_\alpha), \end{cases} \tag{1}$$

in a neighborhood of the origin for some $C > C_*$, then (P) ha NO local-in-time solution.

If we can prove the existence of solution when μ has the singularity as in (1), we see that (1) is the strongest singularity for the solvability of the Cauchy problem (P). In this talk, we also show the existence results. These imply that (1) is the strongest singularity of the inhomogeneous term for the solvability of the Cauchy problem (P).

References

[1] G. Bernard, Existence theorems for certain elliptic and parabolic semilinear equations, J. Math. Anal. Appl. **210** (1997), 755–776.

[2] H. Hisa, K. Ishige and J. Takahashi, Existence os solutions for an inhomogeneous fractional semilinear heat equation, submitted, arXiv:1910.12013v1.

[3] Qi S. Zhang, A new critical phenomenon for semilinear parabolic problems, J. Math. Anal. Appl. **219** (1998), 125–139.

Singularities of solutions for the heat equation with a dynamic Hardy potential

Eiji Yanagida

Tokyo Institute of Technology

We consider the asymptotic properties of singularities of solutions for the heat equation with a time-dependent potential. Assuming that the potential has an inverse-square leading term, we show that there are two types of solutions with different asymptotics that are generic in some sense. For the existence, we first find approximate solutions and then construct suitable comparison functions by modifying singular solutions of the linear heat equation. For the classification, we transform the equation into an integral equation and then estimate the integral. We also show the existence of a solution in the critical case, and study its properties. This talk is based on a joint work with Jin Takahashi.

Around concavity properties and heat transfer

Paolo Salani
Universitá di Firenze

Concavity properties of solutions of PDEs are a popular subject of investigation. Among them, power concavities have been the most studied and log-cocavity is probably the most famous, due to its relevance to many applied disciplines (especially economics and statistics) and its connection with heat transfer. A classical result by Brascamp and Lieb indeed says that the log-concavity of the initial datum is preserved by the heat flow in a convex domain, and this has always been considered an optimal property. Recently, in a joint work with K. Ishige and A. Takatsu, we proved that actually a stronger concavity is preserved by the heat transfer in a natural way and we introduced a new family of concavity properties that we call ”power log-concavities”. I will present some connected results and some characterizations of log-concavity.

Elliptic and parabolic boundary value problems on rotationally symmetric domains

Asuka Takatsu
Tokyo Metropolitan University

We study power concavity of solutions to elliptic and parabolic boundary value problems on rotationally symmetric, strongly convex open metric balls in a Riemannian manifold. Our results provide a first step to the study of power concavity for Riemannian manifolds, and improve the known results for Euclidean spaces. This is joint work with Kazuhiro Ishige (The University of Tokyo) and Paolo Salani (University of Florence).

Blow-up of radially symmetric solutions for a semilinear heat equation on hyperbolic space

Masahiko Shimojo
Okayama University of Science

In this talk radially symmetric solutions of a semilinear heat equation $u_t = \Delta u + u^p$ on the hyperbolic space are considered. First we analyze blow-up set of solutions and derive its local blow-up profile when the origin is not a blow-up point. Next the universal bounds of the nonnegative solutions are obtained together with the local blwo-up profile, under the assumption that the exponent p is subcritical in the Sobolev sense. This is the joint work with Ai Ling Poh.

Optimal singularities of initial functions for the solvability of a semilinear parabolic system

Yohei Fujishima
Shizuoka University

We study the solvability for a weakly coupled system of semilinear heat equations

$$\begin{cases} \partial_t u = D_1 \Delta u + v^p, & x \in \mathbf{R}^N, t > 0, \\ \partial_t v = D_2 \Delta v + u^q, & x \in \mathbf{R}^N, t > 0, \\ (u(\cdot,0), v(\cdot,0)) = (\mu, \nu), & x \in \mathbf{R}^N, \end{cases} \tag{P}$$

where D_1, $D_2 > 0$, $0 < p \le q$ with $pq > 1$ and (μ, ν) is a pair of nonnegative measurable functions in $\mathbf{R}^N$. We consider the existence and nonexistence of solutions for problem (P) with singular initial functions μ and ν. In particular, we derive necessary conditions of problem (P), which enable us to identify optimal singularities of the initial functions for the solvability of problem (P).

This talk is based on a joint work with K. Ishige (The University of Tokyo, Japan).